Math 125B
Vogler

Properties for Multiple Integrals on Jordan Regions

Thm (12.23) Linear Properties

Let E be a Jordan region in $\mathbb{R}^n$, $f,g: E \to \mathbb{R}$, and $\alpha \in \mathbb{R}$.

1) If $f,g \in \mathcal{R}(E)$, then $f+g \in \mathcal{R}(I)$ and $\alpha f \in \mathcal{R}(I)$. Moreover,
$$\int_E \alpha f(\vec{x})\,d\vec{x} = \alpha \int_E f(\vec{x})\,d\vec{x} \text{ and } \int_E (f+g)\,d\vec{x} = \int_E f(\vec{x})\,d\vec{x} + \int_E g(\vec{x})\,d\vec{x}.$$

2) If $E_1, E_2 \subseteq E$ are nonoverlapping Jordan regions and $f \in \mathcal{R}(E_1)$ and $f \in \mathcal{R}(E_2)$, then $f \in \mathcal{R}(E_1 \cup E_2)$ and
$$\int_{E_1 \cup E_2} f(\vec{x})\,d\vec{x} = \int_{E_1} f(\vec{x})\,d\vec{x} + \int_{E_2} f(\vec{x})\,d\vec{x}.$$

-Note: Compare 2) to its 1 variable counterpart Thm 5.20.

Thm (12.24)

Let E be a Jordan region in $\mathbb{R}^n$, $E_0 \subseteq E$, and $f,g: E \to \mathbb{R}$ be bounded.

1) If E_0 has zero volume, then $f \in \mathcal{R}(E_0)$ and $\int_{E_0} f(\vec{x})\,d\vec{x} = 0$.

2) If $f \in \mathcal{R}(E)$, E_0 has zero volume, and $g(\vec{x}) = f(\vec{x})\ \forall \vec{x} \in E - E_0$, then $g \in \mathcal{R}(E)$ and
$$\int_E g(\vec{x})\,d\vec{x} = \int_E f(\vec{x})\,d\vec{x}.$$

-Note: Since ∂E has zero volume, this tells us $\int_{\partial E} f(\vec{x})\,d\vec{x} = 0$ for all bounded functions f, so boundries never contribute to the value of the integral of bounded functions.

Thm (12.25) Comparison Theorem for Multiple Integrals

Let $E \subseteq \mathbb{R}^n$ be a Jordan region and $f,g: E \to \mathbb{R}$ where $f,g \in \mathcal{R}(E)$. Then, we have

1) If $f(\vec{x}) \leq g(\vec{x})$ $\forall \vec{x} \in E$, then
$$\int_E f(\vec{x})d\vec{x} \leq \int_E g(\vec{x})d\vec{x}.$$

2) If $m, M \in \mathbb{R}$ such that $m \leq f(\vec{x}) \leq M$ $\forall \vec{x} \in E$, then
$$m \, Vol(E) \leq \int_E f(\vec{x})d\vec{x} \leq M \, Vol(E).$$

3) $|f| \in \mathcal{R}(E)$ and $\left|\int_E f(\vec{x})d\vec{x}\right| \leq \int_E |f(\vec{x})|d\vec{x}$.

Thm (12.26) Mean Value Theorem for Multiple Integrals

Let $E \subseteq \mathbb{R}^n$ be a Jordan region and $f,g: E \to \mathbb{R}$ with $f,g \in \mathcal{R}(E)$ and $g(\vec{x}) \geq 0$ $\forall \vec{x} \in E$. Then, we have

1) There is a number $c \in \mathbb{R}$ which satisfies
$$\inf_{\vec{x} \in E} f(\vec{x}) \leq c \leq \sup_{\vec{x} \in E} f(\vec{x}) \quad (*)$$
such that
$$c \int_E g(\vec{x})d\vec{x} = \int_E f(\vec{x}) g(\vec{x}) d\vec{x}.$$

2) There is a number $c \in \mathbb{R}$ which satisfies $(*)$ where
$$c \, Vol(E) = \int_E f(\vec{x}) d\vec{x}.$$